FIRST AND SECOND SEMESTER B.Tech DEGREE MODEL EXAM

EN14 102 ENGINEERING MATHEMATICS II

TIME:3 Hrs **MAX:100 MARKS**

Part A(Answer any EIGHT questions)

I. 1. Solve: $\frac{dy}{dx} = \frac{x+y}{x-y}$

2.Solve: $\frac{dy}{dx} + y\, secx = tanx.$

3.Solve: $(x+y+1)^2 \frac{dy}{dx} = 1$

4. Find the Laplace transform of (i) $sinh^2 t$ (ii) $\frac{1-cost}{t}$.

5. Find the inverse Laplace transform of $\frac{s}{(s^2+a^2)^2}$.

6. Evaluate $L^{-1}\left[\frac{1+3s}{(s-1)(s^2+1)}\right]$.

7. If $r = |\overrightarrow{r}|$ where $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ prove that (i) $\nabla r^n = nr^{n-2}\vec{r}$
(ii) $\nabla\,(\mathrm{a.r}) = a.$

8. Show that $\vec{A} = (x^2 - y^2 + x)\hat{\imath} - (2xy + y)\hat{\jmath}$ is irrotational and hence find the scalar potential φ.

9. Evaluate $\iint_R y\,dxdy$ over the area between $y^2 = 4x$ and $x^2 = 4y$.

10. Evaluate $\int_0^a \int_0^a \int_0^a (yz + zx + xy)\, dxdydz$ $(8 \times 5 = 40 Marks)$

Part B

II.(A) Solve(1) $\frac{dy}{dx} = \frac{-7x+3y+7}{3x-7y-3}$ (2) $(x^2 - 4xy - 2y^2)dx + (y^2 - 4xy - 2x^2)\, dy = 0$

$(15\ Marks)$

OR

(B). Find the orthogonal trajectories of (1) cardiods $r = \alpha(1 + \cos\theta)$
(2) parabolas $y^2 = 4a(x + a)$ $(15\ Marks)$

III.(A). (i)Evaluate $L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right]$ using convolution theorem. $(8\ Marks)$

(ii) Evaluate $L^{-1}\left[\frac{2s+5}{s^2+4s-5}\right]$. $(7\ Marks)$

OR

(B). (i) Solve $y'' + 2y' - 3y = sint$, when $y(0) = 0\ and\ y'(0) = 0$ using Laplace transforms. $(8\ Marks)$

(ii) Find the Laplace transform of $f(t) = \begin{cases} t, 0 \le t \le c \\ 2c - t, c < t \le 2c \end{cases}$ where $f(t + 2c) = f(t)$ for all t . $(7\ Marks)$

IV.(A).If $u = x^2yz$ and v= $xy - 3z^2$ find $\nabla \times (\nabla u \times \nabla \text{v})$ at the pont (1,1,0). ($15 marks$)

OR

(B).Find the constants a, b, c so that

$\vec{F} = (x + 2y + az)\hat{\imath} + (bx - 3y - z)\hat{\jmath} + (4z + cy + 2z)\hat{k}$ is irrotational.(15 $Marks$).

V.(A) .Verify Divergence theorem for $\vec{F} = x^2\hat{\imath} + y^2\hat{\jmath} + z^2\hat{k}$ taken over the cube bounded by the planes $x = 0, x = 1, y = 0, y = 1, z = 0 , z = 1$.(15 $Marks$)

OR

(B) Verify Green's theorem in the plane for $\oint_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$ where C Is the closed curve of the region bounded by $y = 0, x = 0, x + y = 1$.(15 $Marks$)

Answer key - Engineering Mathematics - II

Part - A

1. Given, $\frac{dy}{dx} = \frac{x+y}{x-y}$ -(1), which is in homogenous form. Hence, put $y = vx$. $\Rightarrow \frac{dy}{dx} = v + x \cdot \frac{dv}{dx}$.

ie;

(1) becomes; $v + x \cdot \frac{dv}{dx} = \frac{x+vx}{x-vx} = \frac{1+v}{1-v}$

$$\Rightarrow x \cdot \frac{dv}{dx} = \frac{1+v^2}{1-v} \Rightarrow \left(\frac{1-v}{1+v^2}\right) dv = \frac{dx}{x}.$$

On integrating; $\int \frac{1}{1+v^2} dv - \int \frac{v}{1+v^2} dv = \int \frac{dx}{x}$.

Put $1+v^2 = u$.
$2v\,dv = du$

$$\Rightarrow \tan^{-1}(v) - \frac{1}{2} \log(u) = \log x + c.$$

$$\Rightarrow 2\tan^{-1}(v) = 2\log x + \log(u) + c.$$

$$= \log x^2 + \log(1+v^2) + c$$

$$= \log\left(x^2 \cdot \left(1 + \frac{y^2}{x^2}\right)\right) + c = \log(x^2+y^2) + c.$$

$$\Rightarrow 2\tan^{-1}(y/x) = \log(x^2+y^2) + c$$

2. $\frac{dy}{dx} + y \sec x = \tan x$; which is linear form.

So, here $I.F = e^{\int P dx} = e^{\int \sec x \, dx} = e^{\log(\sec x + \tan x)}$

$$= \sec x + \tan x.$$

$\therefore$ Soln is $y(I.F) = \int Q.(I.F) + c.$

$$y(\sec x + \tan x) = \int \tan x (\sec x + \tan x)\, dx + c$$

$$= \int \sec x \tan x \, dx + \int \tan^2 x \, dx + c.$$

$$= \sec x + \int (\sec^2 x - 1)\, dx + c$$

$$= \sec x + \tan x - x + c.$$

③. Given $(x+y+1)^2 \frac{dy}{dx} = 1$ —(1)

Put $x+y+1 = t \Rightarrow 1 + \frac{dy}{dx} = \frac{dt}{dx}$.

$\therefore$ (1) becomes; $t^2\left(\frac{dt}{dx} - 1\right) = 1 \Rightarrow \frac{dt}{dx} = \frac{1+t^2}{t^2}$

$$\Rightarrow \int \frac{t^2}{1+t^2}\, dt = \int dx \Rightarrow \int 1 - \frac{1}{1+t^2}\, dt = \int dx.$$

$$\Rightarrow t - \tan^{-1}(t) = x + c \Rightarrow (x+y+1) - \tan^{-1}(x+y+1) = x+c$$

$$\Rightarrow y = \tan^{-1}(x+y+1) + c - 1$$
$$= \tan^{-1}(x+y+1) + C \text{, where } C = c-1.$$

④ (i) $\mathcal{L}(\sinh^2 t) = \mathcal{L}\left(\left(\frac{e^t - e^{-t}}{2}\right)^2\right)$

$$= \mathcal{L}\left(\frac{e^{2t} - 2 + e^{-2t}}{4}\right) = \frac{1}{4}\left[\frac{1}{s-2} - 2\cdot\frac{1}{s} + \frac{1}{s+2}\right]$$

$$= \frac{1}{4}\left[\frac{2s}{s^2-4} - \frac{2}{s}\right] = \frac{1}{4}\left[\frac{8}{s(s^2-4)}\right] = \frac{2}{s(s^2-4)}$$

(ii) $\mathcal{L}\left(\frac{1-\cos t}{t}\right) = \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+1}\right) ds.$

$f(t) = 1 - \cos t$
$\bar{f}(s) = \mathcal{L}(f(t))$
$= \frac{1}{s} - \frac{s}{s^2+1}$

$$= \left[\log s - \frac{1}{2}\log(s^2+1)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\frac{s^2}{s^2+1}\right]_s^\infty = \frac{1}{2}\left[\lim_{s\to\infty} \log\left(\frac{s^2}{s^2+1}\right) - \log\left(\frac{s^2}{s^2+1}\right)\right]$$

$$= \frac{1}{2}\left[\log 1 - \log\frac{s^2}{s^2+1}\right] = \frac{1}{2}\log\left(\frac{s^2+1}{s^2}\right)$$

⑤ $\mathcal{L}^{-1}\left(\frac{s}{(s^2+a^2)^2}\right)$.

Use convolution theorem. Here, $\bar{f}(s) = \frac{s}{s^2+a^2}$, $\bar{g}(s) = \frac{1}{s^2+a^2}$

$f(t) = \cos at$; $g(t) = \frac{1}{a}\sin at$

Hence, $\mathcal{L}^{-1}\left(\frac{s}{(s^2+a^2)^2}\right) = \mathcal{L}^{-1}\left(\frac{s}{s^2+a^2}\cdot\frac{1}{s^2+a^2}\right) = \int_0^t \cos(a(t-u))\cdot\frac{\sin au}{a}\,du$

$$= \frac{1}{2a}\int_0^t 2\sin au\cdot\cos a(t-u)\,du$$

$$= \frac{1}{2a}\int_0^t [\sin at + \sin(2au-at)]\,du$$

$$= \frac{1}{2a}\left[u\sin at - \frac{1}{2a}\cos(2au-at)\right]_0^t = \frac{1}{2a}t\sin at.$$

(6) $\mathcal{L}^{-1}\left(\frac{1+3s}{(s-1)(s^2+1)}\right)$

$$\frac{1+3s}{(s-1)(s^2+1)} = \frac{A}{(s-1)} + \frac{Bs+C}{s^2+1}$$

Putting $s=1$, $A=2$.

Equating coef of s^2, $0 = A+B \Rightarrow B=-A$
$\Rightarrow B = -2$

Putting $s=0$; $C=1$

$$\therefore \mathcal{L}^{-1}\left(\frac{1+3s}{(s-1)(s^2+1)}\right) = \mathcal{L}^{-1}\left[\frac{2}{(s-1)} + \frac{-2s+1}{(s^2+1)}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{2}{(s-1)} + -2\cdot\frac{s}{s^2+1} + \frac{1}{s^2+1}\right]$$

$$= 2e^t - 2\cdot\cos t + \sin t$$

(7) (i) given $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$. Hence; $\nabla r = \frac{\vec{r}}{r}$

$$\nabla r^n = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)r^n = \frac{\partial r^n}{\partial x}\hat{i} + \frac{\partial r^n}{\partial y}\hat{j} + \frac{\partial r^n}{\partial z}\hat{k}$$

$$= nr^{n-1}\frac{\partial r}{\partial x}\hat{i} + nr^{n-1}\frac{\partial r}{\partial y}\hat{j} + nr^{n-1}\frac{\partial r}{\partial z}\hat{k} = nr^{n-1}(\nabla r)$$

$$= nr^{n-1}\cdot\frac{\vec{r}}{r} = nr^{n-2}\vec{r}$$

(ii) $\nabla(\vec{a}\cdot\vec{r}) = \nabla(a_1x + a_2y + a_3z)$ $\left(\because \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k},\ \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}\right)$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(a_1x + a_2y + a_3z)$$

$$= a_1\hat{i} + a_2\hat{j} + a_3\hat{k} = \vec{a}.$$

(8) $\vec{A} = (x^2 - y^2 + x)\hat{i} - (2xy + y)\hat{j}$.

Here (find) curl $\vec{A} = \vec{0}$. Hence $\vec{A}$ is irrotational.

$\therefore \vec{A} = \nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$.

$\Rightarrow \frac{\partial\phi}{\partial x} = x^2 - y^2 + x$. and $\frac{\partial\phi}{\partial y} = -(2xy + y)$ and $\frac{\partial\phi}{\partial z} = 0$.

$\Downarrow$

$\Rightarrow \phi = \frac{x^3}{3} - xy^2 + \frac{x^2}{2} + \phi_1(y,z)$ —(1)

$\phi = -\left(xy^2 + \frac{y^2}{2}\right) + \phi_2(x,z)$ —(2)

$\phi = \phi_3(x,y)$

Here, $\phi_1(y,z) =$ terms indpt of 'x' $= -\frac{y^2}{2}$

$\phi_2(x,z) =$ terms indpt of 'y' $= \frac{x^3}{3} + \frac{x^2}{2}$

$\phi_3(x,y) =$ terms indpt of 'z' $= \frac{x^3}{3} + \frac{x^2}{2} - xy^2 - \frac{y^2}{2}$

$\therefore \phi = \frac{x^3}{3} + \frac{x^2}{2} - xy^2 - \frac{y^2}{2}$

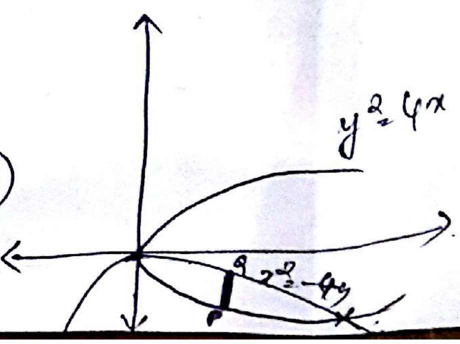
(9) Here the region is between $y^2 = 4x$ and $x^2 = -4y$.

$\therefore$ Point of intersection:

$y^2 = 4x \Rightarrow y^4 = 16x^2$

$\Rightarrow y^4 = 16(-4y) = -64y$

$\Rightarrow y^4 + 64y = 0 \Rightarrow y(y^3 + 64) = 0$



$y=0, \quad y=-4.$

$x=0,$ and $x=4$

$\therefore$ point of intersection are $(0,0)$ and $(4,-4)$.

$$\therefore \iint_R y\,dx\,dy = \int_0^4 \int_{2\sqrt{x}}^{-x^2/4} y\,dy\,dx = \frac{-48}{5}$$

(10) $$\int_0^a \int_0^a \int_0^a (yz + 3x + xy)\,dx\,dy\,dz$$

$$= \int_0^a \int_0^a \left[xyz + \frac{3x^2}{2} + \frac{x^2 y}{2}\right]_0^a dy\,dz$$

$$= \int_0^a \int_0^a \left(ayz + \frac{a^2 z}{2} + \frac{a^2 y}{2}\right) dy\,dz$$

$$= \int_0^a \left[\frac{ay^2 z}{2} + \frac{a^2 yz}{2} + \frac{a^2 y^2}{4}\right]_0^a dz$$

$$= \int_0^a \left[\frac{a^3 z}{2} + \frac{a^3 z}{2} + \frac{a^4}{4}\right] dz$$

$$= \int_0^a \left(a^3 z + \frac{a^4}{4}\right) dz = \left(\frac{a^3 z^2}{2} + \frac{a^4 z}{4}\right)_0^a$$

$$= \frac{a^5}{2} + \frac{a^5}{4} = \frac{3a^5}{4}$$

Part B

II (A) (1) Here $\dfrac{dy}{dx} = \dfrac{-7x+3y+7}{3x-7y-3}$ — ①

Put $x = X+h,\ y = Y+k$ in ①

$\therefore \dfrac{dY}{dX} = \dfrac{-7(X+h)+3(Y+k)+7}{3(X+h)-7(Y+k)-3}$ — (2)

Choose h and k such that $-7h+3k+7=0$

$3h-7k-3=0$

Solving these $h = \dfrac{\begin{vmatrix} 3 & 7 \\ -7 & -3 \end{vmatrix}}{\begin{vmatrix} -7 & 3 \\ 3 & -7 \end{vmatrix}} = \dfrac{-9+49}{49-9} = \underline{\underline{1}}$

$k = \dfrac{\begin{vmatrix} 7 & -7 \\ -3 & 3 \end{vmatrix}}{\begin{vmatrix} -7 & 3 \\ 3 & -7 \end{vmatrix}} = \dfrac{21-21}{49-9} = 0$

So $h=1,\ k=0$ with these values (2) becomes

$\dfrac{dY}{dX} = \dfrac{-7X+3Y}{3X-7Y}$ — (3)

Put $Y = VX \Rightarrow \dfrac{dY}{dX} = V + X\dfrac{dV}{dX}$

$\therefore$ (3) becomes $V + X\dfrac{dV}{dX} = \dfrac{-7X+3Y}{3X-7Y}$

$\Rightarrow V + X\dfrac{dV}{dX} = \dfrac{-7X+3VX}{3X-7VX}$

$$\Rightarrow V + X\frac{dV}{dX} = \frac{-7+3V}{3-7V}$$

$$\Rightarrow \frac{3-7V}{(V^2-1)}dV = 7\frac{dX}{X}$$

$$\Rightarrow \frac{3-7V}{(1-V^2)} = -7\frac{dX}{X}$$

$$\Rightarrow \left(\frac{5}{1+V} - \frac{2}{1-V}\right)dV = -7\frac{dX}{X}$$

On integnation

$$5\log(1+V) - {}^{-}2\log(1-V) = -7\log X + \log c$$

$$\Rightarrow \log(1+V)^5(1-V)^2 + \log X^7 = \log c$$

$$\Rightarrow (1+V)^5(1-V)^2X^7 = C$$

$$\Rightarrow \left(1+\frac{Y}{X}\right)^5\left(1-\frac{Y}{X}\right)^2 X^7 = C$$

$$\Rightarrow (X+Y)^5(X-Y)^2 = C$$

but $x = X+1$, and $y = Y+0$

$\Rightarrow X = x-1$ and $Y = y$

$$\therefore (x-1+y)^5(x-1-y)^2 = C.$$

(2) $(x^2 - 4xy - 2y^2)\,dx + (y^2 - 4xy - 2x^2)\,dy = 0$

Comparing with the standard form $Mdx + Ndy = 0$

$M = x^2 - 4xy - 2y^2$ and $N = y^2 - 4xy - 2x^2$

$\frac{\partial M}{\partial y} = -4x - 4y$. $\frac{\partial N}{\partial x} = -4y - 4x$ hence the equation is exact.

$\therefore$ the solution is

$$\int (x^2 - 4xy - 2y^2)\,dx + \int y^2 dy = c$$

$$\Rightarrow \frac{x^3}{3} - 2x^2y - 2y^2x + \frac{y^3}{3} = c$$

$$\Rightarrow x^3 - 6x^2y - 6xy^2 + y^3 = c$$

OR

(B)

(1) Given $r = \alpha(1+\cos\theta)$ —— ①

differentiating this $\frac{dr}{d\theta} = -\alpha \sin\theta$

$$\Rightarrow \alpha = -\frac{1}{\sin\theta}\frac{dr}{d\theta}$$

Substitute in ① $r = -\frac{1}{\sin\theta}\frac{dr}{d\theta}(1+\cos\theta)$

$$\Rightarrow \frac{dr}{d\theta} = -r\cdot\frac{\sin\theta}{1+\cos\theta}$$

Replace $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$

$$\text{i.e. } -r^2\frac{d\theta}{dr} = -r\cdot\frac{\sin\theta}{1+\cos\theta}$$

$$\Rightarrow \frac{1+\cos\theta}{\sin\theta}\,d\theta = \frac{dr}{r}$$

$$\Rightarrow \frac{\cos\theta/2}{\sin\theta/2}\,d\theta = \frac{dr}{r}$$

on integration $2\log\sin\theta/2 = \log rk$

$\Rightarrow \sin^2 \theta/_2 = rk$

$\Rightarrow \left(\frac{1-\cos\theta}{2}\right) = rk$

$\Rightarrow r = \beta(1-\cos\theta)$ where $\beta = \frac{1}{2k}$.

(2.) Given $y^2 = 4a(x+a)$ — ①

differentiate ① $2y\frac{dy}{dx} = 4a$

$\Rightarrow a = \frac{y}{2}\frac{dy}{dx}$

Substitute in ① $y^2 = 4\frac{y}{2}\frac{dy}{dx}\left(x + \frac{y}{2}\frac{dy}{dx}\right)$

$\Rightarrow y = 2xy\frac{dy}{dx} + y^2\left(\frac{dy}{dx}\right)^2$

$\Rightarrow y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - y = 0$ — (2) which is

the differential equation of the given family.

Replacing $\frac{dy}{dx}$ by $-\frac{1}{\left(\frac{dy}{dx}\right)}$

$y\left(\frac{dy}{dx}\right)^2 + 2x\left(\frac{dy}{dx}\right) - y = 0$ — (3)

Hence the given system of curves is self orthogonal.

III (A) (1.) $L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = L^{-1}\left[\frac{1}{s} \cdot \frac{s}{(s^2+a^2)^2}\right]$

Let $F(s) = \frac{1}{s}$, $G(s) = \frac{s}{(s^2+a^2)^2}$

Then $f(t) = L^{-1}\left[\frac{1}{s}\right] = 1$, $g(t) = L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$

$= \frac{1}{2a} t \sin at$

Hence $L^{-1}\left[\frac{1}{s}\cdot\frac{s}{(s^2+a^2)^2}\right] = \frac{1}{2a}\int_0^t 1.\ u \sin au\, du$

$$= \frac{1}{2a}\left[-u\frac{\cos au}{a} + \int_0^t \frac{\cos au}{a}du\right]_0^t$$

$$= \frac{1}{2a}\left[-\frac{t\cos at}{a} + \left(\frac{\sin au}{a^2}\right)_0^t\right]$$

$$= \frac{1}{2a}\left[-\frac{t\cos at}{a} + \frac{\sin at}{a^2}\right]$$

$$= \frac{1}{2a^3}\left[\sin at - at\cos at\right].$$

(2) $L^{-1}\left[\frac{2s+5}{s^2+4s-5}\right] = L^{-1}\left[\frac{2(s+2)+1}{(s+2)^2-9}\right]$

$$= 2L^{-1}\left[\frac{s+2}{(s+2)^2-9}\right] + L^{-1}\left[\frac{1}{(s+2)^2-9}\right]$$

$$= \left(2.\ e^{-2t}\cosh 3t + \frac{e^{-2t}\sinh 3t}{3}\right)$$

$$= e^{-2t}\left(2\cosh 3t + \tfrac{1}{3}\sinh 3t\right).$$

OR

(B) (1). Given $y'' + 2y' - 3y = \sin t$

Taking Laplace transform on both sides

$$L[y'' + 2y' - 3y] = L[\sin t]$$

$$\Rightarrow [s^2 Y(s) - s y(0) - y'(0)] + 2[s Y(s) - y(0)] - 3Y(s) = \frac{1}{s^2+1}$$

$$\Rightarrow (s^2 + 2s + 3)Y(s) = \frac{1}{s^2+1}$$

$$\Rightarrow Y(s) = \frac{1}{(s^2+1)(s^2+2s-3)}$$

$$= \frac{1}{(s-1)(s+3)(s^2+1)}$$

$$= \frac{1/8}{s-1} + \frac{-1/40}{s+3} + \frac{-1}{10}\frac{(s+1)}{s^2+1}$$ (by partial fractions)

$$\therefore y(t) = \frac{1}{8}L^{-1}\left[\frac{1}{s-1}\right] - \frac{1}{40}L^{-1}\left[\frac{1}{s+3}\right] - \frac{1}{10}L^{-1}\left[\frac{s}{s^2+1}\right] - \frac{1}{10}L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$\therefore y(t) = \frac{1}{8}e^{t} - \frac{1}{40}e^{-3t} - \frac{1}{10}\cos t - \frac{1}{10}\sin t$$

(2) Given $f(t) = \begin{cases} t, & 0 \le t \le c \\ 2c-t, & c \le t \le 2c \end{cases}$

$$L[f(t)] = \frac{1}{1-e^{-2cs}}\int_0^{2c} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2cs}}\left[\int_0^c t e^{-st}\,dt + \int_c^{2c}(2c-t)e^{-st}\,dt\right]$$

$$= \frac{1}{1-e^{-2cs}}\left\{\left[t\cdot\frac{e^{-st}}{-s} - \int_0^c \frac{e^{-st}}{-s}\,dt\right]_0^c + \left[(2c-t)\frac{e^{-st}}{-s} - \int_c^{2c}(-1)\frac{e^{-st}}{-s}\,dt\right]_c^{2c}\right\}$$

$$= \frac{1}{1-e^{-2cs}}\left\{\frac{ce^{-cs}}{-s} + \frac{1}{s}\left[\frac{e^{-st}}{-s}\right]_0^c + \frac{ce^{-cs}}{s} - \frac{1}{s}\left[\frac{e^{-st}}{-s}\right]_c^{2c}\right\}$$

$$= \frac{1}{1-e^{-2cs}}\left\{ \frac{-c\,e^{-cs}}{s} - \frac{1}{s^2}\left(e^{-cs}-1\right) + \frac{c e^{-cs}}{s} + \frac{1}{s^2}\left(e^{-2cs} - e^{-cs}\right)\right\}$$

$$= \frac{\left(1-e^{-cs}\right)^2}{s^2\left(1-e^{-2cs}\right)}$$

IV (A) Given $u = x^2yz$ and $v = xy - 3z^2$

$$\nabla u = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2yz)$$

$$= 2xyz\,\hat{i} + x^2z\,\hat{j} + x^2y\,\hat{k}$$

$$\nabla v = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(xy - 3z^2)$$

$$= y\,\hat{i} + x\,\hat{j} - 6z\,\hat{k}$$

$$\nabla u \times \nabla v = \begin{vmatrix} i & j & k \\ 2xyz & x^2z & x^2y \\ y & x & -6z \end{vmatrix}$$

$$= (-6x^2z^2 - x^3y)\,\hat{i} + (12xyz^2 + x^2y^2)\,\hat{j} + x^2yz\,\hat{k}$$

$$\therefore \nabla \times (\nabla u \times \nabla v) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -6x^2z^2 - x^3y & 12xyz^2 + x^2y^2 & x^2yz \end{vmatrix}$$

$$= \frac{1}{1-e^{-2cs}}\left\{\frac{-c\,e^{-cs}}{s} - \frac{1}{s^2}(e^{-cs}-1) + \frac{c e^{-cs}}{s} + \frac{1}{s^2}(e^{-2cs} - e^{-cs})\right\}$$

$$= \underline{\underline{\frac{(1-e^{-cs})^2}{s^2(1-e^{-2cs})}}}.$$

IV (A) Given $u = x^2yz$ and $v = xy - 3z^2$

$$\nabla u = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2yz)$$

$$= 2xyz\,\hat{i} + x^2z\,\hat{j} + x^2y\,\hat{k}$$

$$\nabla v = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(xy - 3z^2)$$

$$= y\,\hat{i} + x\,\hat{j} - 6z\,\hat{k}$$

$$\nabla u \times \nabla v = \begin{vmatrix} i & j & k \\ 2xyz & x^2z & x^2y \\ y & x & -6z \end{vmatrix}$$

$$= (-6x^2z^2 - x^3y)\,\hat{i} + (12xyz^2 + x^2y^2)\,\hat{j} + x^2yz\,\hat{k}$$

$$\therefore \nabla \times (\nabla u \times \nabla v) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -6x^2z^2 - x^3y & 12xyz^2 + x^2y^2 & x^2yz \end{vmatrix}$$

$$= (x^2z - 24xyz)\hat{i} - (2xyz + 12x^2z)\hat{j} + (12yz^2 + 2xy^2 + x^3)\hat{k}$$

at $(1,1,0)$ $\nabla \times (\nabla u \times \nabla v) = \underline{3\hat{k}}$.

OR

(13) Given $(x+2y+az)\hat{i} + (bx-3y-z)\hat{j} + (4x-cy+2z)\hat{k}$ is irrotational.

Hence $\nabla \times [(x+2y+az)\hat{i} + (bx-3y-z)\hat{j} + (4x-cy+2z)\hat{k}] = 0$

i.e.
$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x-cy+2z \end{vmatrix} = 0$$

$$\Rightarrow \hat{i}\left(\frac{\partial}{\partial y}(4x-cy+2z) - \frac{\partial}{\partial z}(bx-3y-z)\right) - \hat{j}\left(\frac{\partial}{\partial x}(4x-cy+2z) - \frac{\partial}{\partial z}(x+2y+az)\right) + \hat{k}\left(\frac{\partial}{\partial x}(bx-3y-z) - \frac{\partial}{\partial y}(x+2y+az)\right) = 0$$

$$\Rightarrow \hat{i}(-c+1) - \hat{j}(4-a) + \hat{k}(b-2) = 0$$

$\Rightarrow \quad -c+1=0$

$-4+a=0$

$b-2=0$

$\Rightarrow \quad c=1$

$a=4$

$\underline{\underline{b=2}}$.

$\underline{V}$ (A) Given $\vec{F} = x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$

To show that $\iiint_V \nabla\cdot\vec{F}\,dV = \iint_S \vec{F}\cdot\hat{n}\,ds$

Now $\nabla\cdot\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(x^2\hat{i} + y^2\hat{j} + z^2\hat{k}\right)$

$= 2(x+y+z)$

$$\therefore \iiint_V \nabla\cdot\vec{F}\,dV = \int_0^1\int_0^1\int_0^1 2(x+y+z)\,dx\,dy\,dz$$

$$= 2\int_0^1\int_0^1\left(\frac{x^2}{2}+yx+zx\right)_0^1 dy\,dz$$

$$= 2\int_0^1\int_0^1\left(\frac{1}{2}+y+z\right)dy\,dz$$

$$= 2\int_0^1\left(\frac{y}{2}+\frac{y^2}{2}+yz\right)_0^1 dz$$

$$= 2\int_0^1\left(\frac{1}{2}+\frac{1}{2}+z\right)dz$$

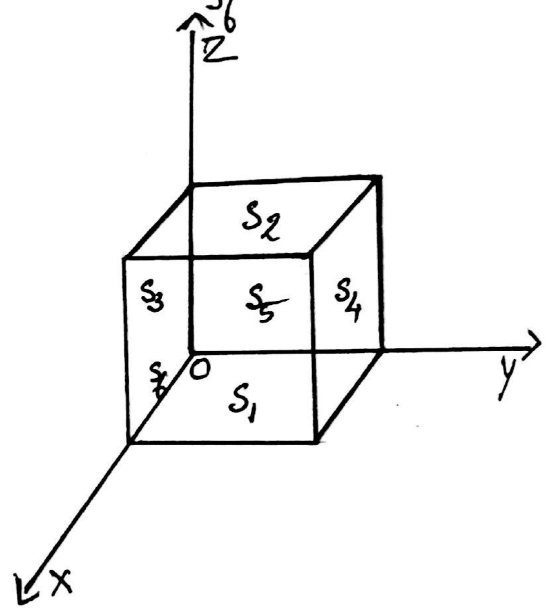
$$= 2\int_0^1(1+z)\,dz$$

$$= 2\left(\cancel{z} + \frac{z^2}{2}\right)_0^1$$

$$= 2(1 + 1/2) = \underline{\underline{3}}.$$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{S_1} \vec{F}\cdot\hat{n}\,ds + \iint_{S_2} \vec{F}\cdot\hat{n}\,ds + \iint_{S_3} \vec{F}\cdot\hat{n}\,ds$$

$$+ \iint_{S_4} \vec{F}\cdot\hat{n}\,ds + \iint_{S_5} \vec{F}\cdot\hat{n}\,ds + \iint_{S_6} \vec{F}\cdot\hat{n}\,ds$$



On S_1: $z = 0$, $\hat{n} = -\hat{k}$

$$\vec{F}\cdot\hat{n} = 0$$

$$\therefore \iint_{S_1} \vec{F}\cdot\hat{n}\,ds = 0$$

On S_2: $z = 1$, $\hat{n} = \hat{k}$

$$\vec{F}\cdot\hat{n} = 1$$

$$\therefore \iint_{S_2} \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 dx\,dy = 1$$

On S_3: $y = 0$, $\hat{n} = -\hat{j}$

$$\vec{F}\cdot\hat{n} = 0$$

$$\therefore \iint_{S_3} \vec{F}\cdot\hat{n}\,ds = 0$$

On S_4: $y = 1$, $\hat{n} = \hat{j}$

$$\vec{F}\cdot\hat{n} = 1$$

$$\therefore \iint_{S_4} \vec{F}\cdot\vec{n}\, ds = \int_0^1\int_0^1 dy\, dz = 1$$

On S_5: $x = 0$, $\vec{n} = -\hat{i}$

$$\vec{F}\cdot\vec{n} = 0$$

$$\therefore \iint_{S_5} \vec{F}\cdot\vec{n}\, ds = 0$$

On S_6: $x = 1$, $\vec{n} = \hat{i}$

$$\vec{F}\cdot\vec{n} = 1$$

$$\therefore \iint_{S_6} \vec{F}\cdot\vec{n}\, ds = \int_0^1\int_0^1 dy\, dz = 1$$

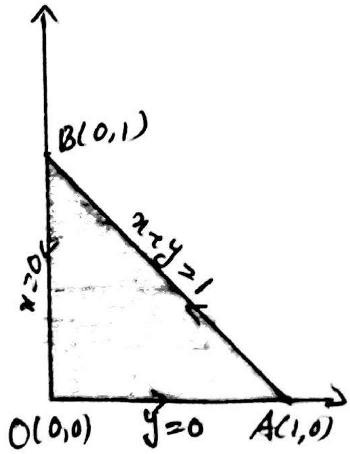
Thus $\iint_S \vec{F}\cdot\vec{n}\, ds = 3$

Hence $\iiint_V \nabla\cdot\vec{F}\, dV = \iint_S \vec{F}\cdot\vec{n}\, ds$

Thus Divergence theorem is verified.

OR

(B) To show that $\oint_C M\, dx + N\, dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$



Here $M = 3x^2 - 8y^2$; $N = 4y - 6xy$

$$\frac{\partial M}{\partial y} = -16y \; ; \; \frac{\partial N}{\partial x} = -6y$$

$$\therefore \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 10y$$

$$\therefore \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy = \int_0^1\int_0^{1-x} 10y\, dy\, dx$$

$$= \int_0^1 5\left(y^2\right)_0^{1-x} dx$$

$$= 5\int_0^1 \left((1-x)^2\right) dx$$

$$= 5\left[\frac{(1-x)^3}{-3}\right]_0^1$$

$$= \underline{\underline{\frac{5}{3}}}$$

Along OA: $y=0$ $\therefore dy=0$ and x varies from 0 to 1

$$\therefore \int_{OA} (3x^2 - 8y^2)dx + (4y - 6xy)dy$$

$$= \int_0^1 3x^2\, dx$$

$$= \left(x^3\right)_0^1 = \underline{\underline{1}}.$$

Along AB: $y = 1-x$ $\therefore dy = -dx$ and x varies from 1 to 0

$$\therefore \int_{AB} (3x^2 - 8y^2)dx + (4y - 6xy)dy$$

$$= \int_1^0 \left[3x^2 - 8(1-x)^2\right]dx + \left[4(1-x) - 6x(1-x)\right](-dx)$$

$$= \int_1^0 (-12 + 26x - 11x^2)dx$$

$$= \left[-12x + 13x^2 - \frac{11}{3}x^3\right]_1^0$$

$$= \underline{\underline{\frac{8}{3}}}$$

Along BO: $x=0$ $\therefore dx=0$ and y varies from 1 to 0

$$\therefore \int_{BO} (3x^2 - 8y^2)dx + (4y - 6xy)dy$$

$$= \int_1^0 4y\,dy$$

$$= \left[2y^2\right]_1^0$$

$$= -2$$

$$\therefore \oint_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$$

$$= 1 + \frac{8}{3} - 2$$

$$= \frac{5}{3}$$

Thus Green's theorem is verified.